PURELY INFINITE SIMPLE C^*-ALGEBRAS ARISING FROM FREE PRODUCT CONSTRUCTIONS

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Abstract. Examples of simple, separable, unital, purely infinite C^*–algebras are constructed, including:

(1) some that are not approximately divisible;
(2) those that arise as crossed products of any of a certain class of C^*–algebras by any of a certain class of non–unital endomorphisms;
(3) those that arise as reduced free products of pairs of C^*–algebras with respect to any from a certain class of states.

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Introduction

We construct three classes of examples of purely infinite, simple, unital C^*–algebras, which may be of special interest. Some of these constructions use Voiculescu's theory freeness and his construction of reduced free products of operator algebras, (see [**14**], see also [**1**]). The first class of examples consists of separable, purely infinite, simple, unital C^*–algebras which are not approximately divisible in the sense of [**2**]. These are the first such examples, and they are constructed by applying a theorem of L. Barnett [**4**] concerning free products of von Neumann algebras, and using an enveloping result proved in this paper. The existence of C^*–algebras with these properties was claimed in [**2**, Example 4.8], but the proposed proof was later seen to be slightly deficient. With Kirchberg's result, which entails that all nuclear, simple, purely infinite C^*–algebras are approximately divisible, (see §1), these examples have become more important and deserving of a complete and correct description.

Skipping ahead, the third class of examples consists of reduced free products of C^*–algebras. Relatively little is understood about the order structure of the K_0 group of reduced free product C^*–algebras, and the knowledge of whether projections in these C^*–algebras are finite or infinite is sporadic. Thus, they are worthy objects of interest, particularly in light of the open question of whether a simple C^*–algebra can be infinite but not purely infinite. We investigate a certain class of reduced free products involving non–faithful states, namely

$$(\mathfrak{A},\varphi)=(A,\varphi_A)*(M_n(\mathbb{C})\otimes B,\varphi_n\otimes\varphi_B),$$

where $A\neq\mathbb{C}$ and B are C^*–algebras with states φ_A and φ_B, and where φ_n is the state on $M_n(\mathbb{C})$ whose support is a minimal projection. We show that $\mathfrak{A}$ can be realized as the $n\times n$ matrices over the crossed product of a C^*–algebra by an endomorphism. We go on to show that, under fairly mild hypotheses, $\mathfrak{A}$ is purely infinite and simple.

Both the second and (as mentioned above) the third class of examples involve crossed products of unital C^*–algebras by non-unital endomorphisms. These are thus in the spirit of Cuntz's presentation of the algebras O_n [**6**]. We give sufficient conditions for such a crossed product to be purely infinite and simple. The second class of examples is the case of the crossed product, $A\rtimes_\sigma\mathbb{N}$, of $A=\bigotimes_1^\infty B$, for B simple and unital, by the endomorphism $\sigma(a_1\otimes a_2\otimes\cdots)=p\otimes a_1\otimes a_2\otimes\cdots$, for $p\in B$ a proper projection. Indeed, the Cuntz algebra O_n is obtained when $B=M_n(\mathbb{C})$ and p is a minimal projection. We show that all such $A\rtimes_\sigma\mathbb{N}$ are purely infinite and simple.

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§1. Non–approximately divisible C^*–algebras

We show in this section that a theorem of L. Barnett implies that there exist purely infinite, simple, separable, unital C^*–algebras that are not approximately divisible. In particular, if A is such a C^*–algebra, then A is not isomorphic to $A \otimes \mathcal{O}_\infty$.

E. Kirchberg has recently proved that if A is a purely infinite, simple, separable, nuclear, unital C^*–algebra, then there exists a sequence of unital *–homomorphisms $\mu_n : \mathcal{O}_\infty \to A$ such that $\mu_n(b)a - a\mu_n(b) \to 0$ as $n \to \infty$ for all $a \in A$ and $b \in \mathcal{O}_\infty$. He concludes from this that A is isomorphic to $A \otimes \mathcal{O}_\infty$ (see [**9**]). Since $\mathcal{O}_\infty$ itself is purely infinite, simple, separable, nuclear and unital, it follows that $\mathcal{O}_\infty$ is isomorphic to $\mathcal{O}_\infty \otimes \mathcal{O}_\infty$. Hence A is isomorphic to $A \otimes \mathcal{O}_\infty$ if and only if A is (isomorphic to) $B \otimes \mathcal{O}_\infty$ for some C^*–algebra B. In [**2**] a unital C^*–algebra A is called approximately divisible if there exists a sequence (or a net, if A is non–separable) of unital *–homomorphisms $\mu_n \colon M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$ such that $\mu_n(b)a - a\mu_n(b) \to 0$ as $n \to \infty$ for all $a \in A$ and $b \in M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$. Since there is a unital embedding of $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ into $\mathcal{O}_\infty$, we conclude from the remarks above that each C^*–algebra, which is isomorphic to $A \otimes \mathcal{O}_\infty$ for some unital C^*–algebra A, is approximately divisible.

Theorem 1.1. (L. Barnett [**4**]). *There is a type III–factor $\mathcal{M}$ with a faithful normal state φ and with elements $a, b, c \in \mathcal{M}$ such that*

$$\|x - \varphi(x) \cdot 1\|_\varphi \le 14 \max\{\|[x,a]\|_\varphi, \|[x,b]\|_\varphi, \|[x,c]\|_\varphi\}$$

for every $x \in \mathcal{M}$.

Corollary 1.2. *Let $\mathcal{M}$ and $a, b, c \in \mathcal{M}$ be as above. Suppose A is a unital C^*–subalgebra of $\mathcal{M}$ which contains a, b and c. Then A is not approximately divisible.*

Proof. Suppose, to reach a contradiction, that A is approximately divisible. Then there is a unital *–homomorphism $\mu : M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A$, such that

$$\max\{\|[a, \mu(x)]\|, \|[b, \mu(x)]\|, \|[c, \mu(x)]\|\} \le \frac{1}{30}\|x\|$$

for all $x \in M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$. There is a projection $e \in M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ such that $1/3 \le \varphi(\mu(e)) \le 1/2$, where φ is the faithful normal state from Theorem 1.1. Indeed, the set Γ of

projections $(p,q) \in M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$, where $\dim(p) = \dim(q) = 1$ is connected. Moreover, there exist $e_1, e_2, e_3 \in \Gamma$ such that $1 = e_1 + e_2 + e_3'$ and $0 \le e_3' \le e_3$, whence $1/3 \le \varphi(\mu(e))$ and $\varphi(\mu(f)) \le 1/2$ for some $e, f \in \Gamma$.

Put $p = \mu(e)$. From Theorem 1.1 we get $\|p - \varphi(p)1\|_\varphi \le 14/30$ since $\|\cdot\|_\varphi \le \|\cdot\|$. On the other hand, since p is a projection, we have

$$\|p - \varphi(p)1\|_\varphi^2 = \varphi(p) - \varphi(p)^2 \ge 2/9,$$

a contradiction. □

Here is the enveloping result mentioned in the introduction.

Proposition 1.3. *Let B be a unital (non–separable) C^*–algebra, and let X be a countable subset of B.*

(i) *If B is simple and purely infinite, then there exists a separable, unital, simple and purely infinite C^*–algebra A such that $X \subseteq A \subseteq B$.*

(ii) *If B is nuclear, then there exists a separable, unital, nuclear C^*–algebra A such that $X \subseteq A \subseteq B$.*

(iii) *If B is simple, purely infinite and nuclear, then there exists a separable, unital, simple, purely infinite and nuclear C^*–algebra A such that $X \subseteq A \subseteq B$.*

Proof. The proofs of (i) and (ii) are easily obtained from the proof given below of (iii).

Suppose B is simple, purely infinite and nuclear. We may assume that $1_B \in X$ so that $X \subseteq A$ will imply that A is unital. Recall that a unital C^*–algebra D is simple and purely infinite if and only if for each positive, non–zero $a \in D$ there exists $x \in D$ with $xax^* = 1$. Moreover, x above can be chosen to have norm less than $2\|a\|^{-1/2}$.

We will show how to construct a sequence $X = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ of countable subsets of B, a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ of separable C^*–subalgebras of B, and a sequence $\Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \cdots$ of countable families of completely positive, finite rank contractions from B into B such that the following four conditions hold for every $n \ge 0$.

(a) For all $\varepsilon \ge 0$ and for each finite subset F of X_n there exists $\varphi \in \Phi_n$ such that $\|\varphi(x) - x\| < \varepsilon$ for all $x \in F$.

(b) $\varphi(X_n) \subseteq A_{n+1}$ for all $\varphi \in \Phi_n$.

(c) X_{n+1} is a dense subset of A_{n+1}, and $X_{n+1} \cap A_{n+1}^+$ is a dense subset of A_{n+1}^+.

(d) For each positive, non–zero $a \in X_n$ there exists an $x \in A_{n+1}$ such that $\|x\| \le 2\|a\|^{-1/2}$ and $xax^* = 1$.

Indeed, given X_n, since there are only countably many finite subsets of X_n, by the Choi–Effros characterization of nuclearity, we can find a countable family Φ_n of completely positive finite rank contractions satisfying (a). Moreover, if $n \geq 1$, then we may insist that $\Phi_n \supseteq \Phi_{n-1}$. For each positive, non–zero element a in B choose $x(a) \in B$ such that $\|x(a)\| \leq 2\|a\|^{-1/2}$ and $x(a)a\,x(a)^* = 1$. Suppose X_n, Φ_n and A_n are given (where $A_0 = \{0\}$). Let A_{n+1} be the C^*–algebra generated by A_n and the countable set

$$\{\varphi(x) \mid x \in X_n \ , \ \varphi \in \Phi_n\} \cup \{x(a) \mid a \in X_n \ , \ a \geq 0, \ a \neq 0\}.$$

Then A_{n+1} is a separable C^*–subalgebra of B, $A_n \subset A_{n+1}$, and (b) and (d) hold. Now choose a countable subset X_{n+1} of B such that $X_n \subseteq X_{n+1}$ and such that (c) holds.

Set

$$A = \overline{\bigcup_{n=1}^{\infty} A_n} \ , \quad Y = \bigcup_{n=0}^{\infty} X_n \ , \quad \Phi = \bigcup_{n=0}^{\infty} \Phi_n .$$

Then Y is a countable dense subset of A and $\varphi(A) \subseteq A$ for all $\varphi \in \Phi$. Hence, by (a), A is a separable, nuclear C^*–subalgebra of B which contains $X \colon (= X_0)$. We must also show that A is simple and purely infinite. Assume $a \in A$ is positive and non–zero. By (c) we can find a (non–zero) positive $a' \in X_n$ for some $n \in \mathbb{N}$ such that $\|a - a'\| \leq \frac{1}{5}\|a\|$. By (d) there exists $y \in A$ such that $ya'y^* = 1$ and $\|y\| \leq 2\|a'\|^{-1/2}$. This implies that

$$\|yay^* - 1\| \leq \|y\|^2 \cdot \|a - a'\| < 1.$$

Hence yay^* is invertible. Set $x = (yay^*)^{-1/2}y \in A$. Then $xax^* = 1$ as desired. □

Theorem 1.4. *There exist C^*–algebras which are separable, unital, simple and purely infinite, but not approximately divisible.*

Proof. Combine Corollary 1.2 and Proposition 1.3 (i) with $B = \mathcal{M}$ and $X = \{a, b, c\}$. Recall from [**7**] that every countably decomposable type III–factor is simple and purely infinite. □

Proposition 1.3 also allows us to sharpen Kirchberg's result, which was discussed at the beginning of this section, on the approximate divisibility of nuclear, simple, purely infinite, unital C^*–algebras. Recall for this that a (non–unital) C^*–algebra A is approximately divisible if A has an approximate unit consisting of projections and if pAp is approximately divisible (in the sense of [**2**]) for all projections p in A.

Theorem 1.5. (cf. Kirchberg [**9**]). *Every nuclear, simple, purely infinite C^*–algebra is approximately divisible.*

Proof. Suppose A is a nuclear, simple, purely infinite C^*–algebra. Then A has real rank zero by [**15**] and so, by [**5**], A has an approximate unit consisting of projections. For each (non–zero) projection p in A, pAp is nuclear, simple and purely infinite.

Let F be a finite subset of pAp. By Proposition 1.3 (iii) there exists a unital, separable, nuclear, simple, purely infinite C^*–subalgebra A_0 of pAp such that $F \subset A_0$. From Kirchberg's theorem [**9**], A_0 is isomorphic to $A_0 \otimes \mathcal{O}_\infty$ and (so) A_0 is approximately divisible. It follows that there, for each $\varepsilon > 0$, exists a unital *–homomorphism $\mu : M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to A_0$ satisfying $\|\mu(b)a - a\mu(b)\| \le \varepsilon \|b\|$ for all $a \in F$ and $b \in M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$. Hence pAp is approximately divisible. □

§2. Crossed products

Associate to each pair consisting of a unital C^*–algebra A and an injective endomorphism σ on A the crossed product $A \rtimes_\sigma \mathbb{N}$, which is the universal C^*–algebra generated by a copy of A and an isometry s such that $sas^* = \sigma(a)$ for all $a \in A$. The isometry s is non–unitary if σ is not unital.

Let $\bar{A}$ be the inductive limit of the sequence

$$A \xrightarrow{\sigma} A \xrightarrow{\sigma} A \xrightarrow{\sigma} \cdots ,$$

and let $\mu_n : A \to \bar{A}$ be the corresponding *–homomorphisms, which satisfy $\mu_{n+1} \circ \sigma = \mu_n$ and

$$\bar{A} = \overline{\bigcup_{n=1}^{\infty} \mu_n(A)}.$$

Observe that $\mu_n : A \to \mu_n(1)\bar{A}\mu_n(1)$ is an isomorphism if and only if σ is a corner endomorphism, i.e. if $\sigma : A \to \sigma(1)A\sigma(1)$ is an isomorphism. If σ is not a corner endomorphism, then A and $\bar{A}$ need not be stably isomorphic.

There is an automorphism α on $\bar{A}$ given by $\alpha(\mu_n(a)) = \mu_n(\sigma(a))$ $(= \mu_{n-1}(a))$ for $a \in A$. The map $\mu_n : A \to \bar{A}$ extends to an isomorphism $\hat{\mu}_n : A \rtimes_\sigma \mathbb{N} \to \mu_n(1)(\bar{A} \rtimes_\alpha \mathbb{Z})\mu_n(1)$ which satisfies $\alpha \circ \hat{\mu}_n = \mu_n \circ \sigma$.

Summarizing results from [**10**] and [**13**] we get the following sufficient conditions to ensure that crossed products by $\mathbb{Z}$ and by $\mathbb{N}$ are purely infinite and simple.

Theorem 2.1.

(i) *Let A be a C^*–algebra, $A \neq \mathbb{C}$, and let α be an automorphism on A. Suppose that α^m is outer for all $m \in \mathbb{N}$ and that A has an approximate unit of projections $(p_n)_1^\infty$ with*

the property that for each $n \in \mathbb{N}$ and for each non–zero hereditary C^*–subalgebra B of A there is a projection in B which is equivalent to $\alpha^m(p_n)$ for some $m \in \mathbb{Z}$. Then $A \rtimes_\alpha \mathbb{Z}$ is purely infinite and simple.

(ii) *Let A be a unital C^*–algebra, $A \neq \mathbb{C}$, and let σ be an injective endomorphism on A. Let α be the automorphism on $\bar{A}$ associated with σ as described above. Suppose that α^m is outer for all $m \in \mathbb{N}$, and suppose that for each non–zero hereditary C^*–subalgebra B of A there is a projection in B which is equivalent to $\sigma^m(1)$ for some $m \in \mathbb{N}$. Then $A \rtimes_\sigma \mathbb{N}$ is purely infinite and simple.*

Proof. (i) By [**13**, Theorem 2.1] and its proof, the claim follows if the conclusions of Lemmas 2.4 and 2.5 in [**13**] hold. Now, [**13**], Lemma 2.4, holds whenever α^m is outer for all $m \in \mathbb{N}$. Secondly, the conclusion of [**13**], Lemma 2.5, is equivalent to the assertion that every non–zero hereditary C^*–subalgebra of A contains a projection, which is infinite relative to the crossed product $A \rtimes_\alpha \mathbb{Z}$. To prove this from the assumptions in (i), it suffices to show that at least one of the projections in the approximate unit $(p_n)_1^\infty$ is infinite in $A \rtimes_\alpha \mathbb{Z}$.

Since $A \neq \mathbb{C}$ there is an $n \in \mathbb{N}$ such that $p_n A p_n \neq \mathbb{C} p_n$. Let B be a non–trivial hereditary C^*–subalgebra of $p_n A p_n$ and let q be a projection in B which is equivalent to $\alpha^m(p_n)$ for an appropriate $m \in \mathbb{Z}$. Because $\alpha^m(p_n)$ is equivalent to p_n in $A \rtimes_\alpha \mathbb{Z}$ and q is a proper subprojection of p_n, we conclude that p_n is infinite.

(ii) Since $A \rtimes_\sigma \mathbb{N}$ is isomorphic to $\mu_1(1)(\bar{A} \rtimes_\alpha \mathbb{Z})\mu_1(1)$, it suffices to show that $\bar{A} \rtimes_\alpha \mathbb{Z}$ is simple and purely infinite. Set $p_{n-m} = \mu_n(\sigma^m(1))$. Then $(p_n)_{n \in \mathbb{Z}}$ is an increasing approximate unit of projections for $\bar{A}$, and $\alpha(p_n) = p_{n-1}$. It suffices, by (i), to show that each non–zero hereditary C^*–subalgebra B of $\bar{A}$ contains a projection equivalent to p_n for some $n \in \mathbb{Z}$. Equivalently, we must show that for each non–zero positive a in $\bar{A}$, we have $p_n = xax^*$ for some $n \in \mathbb{Z}$ and some $x \in \bar{A}$. Find $m \in \mathbb{N}$ and b in A^+ such that $\|\mu_m(b) - a\| \leq \frac{1}{2}\|a\|$. It follows from [**12**], 2.2 and 2.4, that $yay^* = \mu_m(c)$ for some non–zero positive c in A and some y in $\bar{A}$. By assumption, $\sigma^k(1) = zcz^*$ for some $k \in \mathbb{N}$ and some $z \in A$. Hence $p_n = xax^*$ when $n = m - k$ and $x = \mu_m(z)y$. □

We shall consider the following more specific example. Let B be a simple, unital C^*–algebra which contains a non–trivial and proper projection p. Set

$$A = \bigotimes_{j=1}^{\infty} B, \tag{1}$$

and let σ be the injective endomorphism on A given by $\sigma(a) = p \otimes a$. In (1), one must take

tensor product norms making A into a C^*–algebra such that σ exists and is injective. This is possible, for example, by using always $\otimes_{\min}$ or always $\otimes_{\max}$.

Theorem 2.2. *With A and σ as above, the crossed product $A \rtimes_\sigma \mathbb{N}$ is simple and purely infinite.*

The theorem is proved in a number of lemmas that verify that the conditions in Theorem 3.1 (ii) hold.

Lemma 2.3. *If α is the automorphism on $\bar{A}$ associated to σ, then α^m is outer for every $m \in \mathbb{N}$.*

Proof. If α^m were inner, then $\alpha^m(a) = a$ for some non–zero $a \in \bar{A}$. As before, set $p_{n-m} = \mu_n(\sigma^m(1))$ and let

$$e_n = 1 \otimes 1 \otimes \cdots \otimes 1 \otimes p \otimes 1 \otimes \cdots \in A,$$

with p in the n'th tensor factor. Then $\|p_n a p_n - a\|$ tends to 0 and $\|\mu_n(1-e_{2n})a\|$ tends to $\|a\|$ as n tends to infinity, and p_{-n} is orthogonal to $\mu_n(1-e_{2n})$ for all n. Because

$$\|p_{n+m} a p_{n+m} - a\| = \|\alpha^m(p_{n+m} a p_{n+m} - a)\| = \|p_n a p_n - a\|,$$

it follows that $a = p_n a p_n$ for all $n \in \mathbb{Z}$. Hence $\mu_n(1-e_{2n})a = 0$ for all $n \in \mathbb{N}$, which entails that $a = 0$, in contradiction with our assumptions. □

For the remaining part of the proof of Theorem 2.2 we need to consider comparison theory for positive elements as described in [**7**], [**3**] and [**12**]. We remind the reader of the basic theory.

Let A be a C^*–algebra and set $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$. For $a, b \in M_\infty(A)^+$ write $a \precsim b$ if there is a sequence (x_n) in $M_\infty(A)$ such that $x_n b x_n^* \to a$. This order relation extends the usual Murray–von Neumann ordering of the projections in $M_\infty(A)$. If $a \in A^+$ and if p is a projection in A, then $p \precsim a$ if and only if p is equivalent to a projection in the hereditary C^*–subalgebra $\overline{aAa}$, which again is the case if and only if $p = xax^*$ for some $x \in A$.

Let $a, b \in M_\infty(A)^+$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$, and let $a \oplus b$ be the element of $M_\infty(A)^+$ obtained by taking direct sum. Put

$$S(A) = M_\infty(A)^+ / \sim\ ,$$

let $\langle a \rangle \in S(A)$ denote the equivalence class containing $a \in M_\infty(A)^+$, set $\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle$ and write $\langle a \rangle \le \langle b \rangle$ if $a \precsim b$. Then $(S(A), +, \le)$ is an abelian preordered semigroup. Let $DF(A)$ be the set of states on $S(A)$, i.e. the set of additive, order preserving functions $d : S(A) \to \mathbb{R}$ such that $\sup\{d(\langle a \rangle) \mid a \in A\} = 1$.

Lemma 2.4. (cf. [**8**, Lemma 4.1]). *If A is a unital simple C^*–algebra, and if $t, t' \in S(A)$ are such that $d(t) < d(t')$ for all $d \in DF(A)$, then $nt \leq nt'$ for some $n \in \mathbb{N}$.*

Proof. The set of dimension functions $DF(A)$ is weakly compact, which entails that

$$(c =) \sup\{d(t)/d(t') \mid d \in DF(A)\} < 1.$$

Find $m, m' \in \mathbb{N}$ such that $c < m'/m < 1$. Then $d(mt) < d(m't')$ for all $d \in DF(A)$. By [**12**], 3.1, this implies that $kmt + u \leq km't' + u$ for some $k \in \mathbb{N}$ and some $u \in S(A)$. Because A is algebraically simple being a simple unital C^*–algebra, every non–zero element of $S(A)$ is an order unit for $S(A)$. It follows that $lk(m - m')t' \geq u$ for some $l \in \mathbb{N}$. Repeated use of the inequality $kmt + u \leq km't' + u$ yields $lkmt + u \leq lkm't' + u$. Hence, if $n = lkm$, then

$$nt \leq lkmt + u \leq lkm't' + u \leq lkm't' + lk(m - m')t' = nt',$$

as desired. □

Lemma 2.5. *Let A be a unital, simple, infinite dimensional C^*–algebra. For each non–zero a in A^+ and for each $m \in \mathbb{N}$ there is a non–zero b in A^+ such that $m\langle b\rangle \leq \langle a\rangle$ in $S(A)$.*

Proof. It suffices to show this in the case where $m = 2$. Since A is infinite dimensional there exist two non–zero mutually orthogonal positive elements a_1 and a_2 in the hereditary subalgebra $\overline{aAa}$. Observe that $\langle a_1\rangle + \langle a_2\rangle = \langle a_1 + a_2\rangle \leq \langle a\rangle$. By [**11**], 3.4, there is a unitary u in A such that

$$u\,\overline{a_1 A a_1}\,u^* \;\cap\; \overline{a_2 A a_2} \neq \{0\}.$$

Let b be a non–zero positive element in this intersection. Then $b \in \overline{a_2 A a_2}$ and $u^* b u \in \overline{a_1 A a_1}$, whence $\langle b\rangle \leq \langle a_1\rangle$ and $\langle b\rangle \leq \langle a_2\rangle$. This implies $2\langle b\rangle \leq \langle a\rangle$. □

Lemma 2.6. *Let A be a C^*–algebra, let A_0 be a C^*–subalgebra of A and suppose that $a, b \in A_0^+$ are such that $n\langle a\rangle \leq n\langle b\rangle$ in $S(A_0)$ for some $n \in \mathbb{N}$. Suppose further that for some $m \in \mathbb{N}$ there is a set of matrix units $(e_{ij})_{1 \leq i,j \leq n}$ in $M_m(A \cap A_0')$ and that there is a projection $e \in A \cap A_0'$ such that*

$$\hat{e} = \begin{pmatrix} e & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \in M_m(A \cap A_0')$$

is equivalent to e_{11} in $M_m(A \cap A_0')$. Then $n\langle ae\rangle \leq m\langle b\rangle$ in $S(A)$.

Proof. Put

$$\bar{a} = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b & & & 0 \\ & b & & \\ & & \ddots & \\ 0 & & & b \end{pmatrix} \in M_n(A_0).$$

Then $\langle \bar{a} \rangle = n\langle a \rangle$ and $\langle \bar{b} \rangle = n\langle b \rangle$, and so $\langle \bar{a} \rangle \leq \langle \bar{b} \rangle$ in $S(A_0)$. It follows that $x_k \bar{b} x_k^* \to \bar{a}$ for some sequence (x_k) in $M_n(A_0)$. Let $x_k(i,j) \in A_0$ be the (i,j)'th entry of x_k, set

$$\tilde{a} = \begin{pmatrix} a & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{pmatrix}, \; \tilde{b} = \begin{pmatrix} b & & & 0 \\ & b & & \\ & & \ddots & \\ 0 & & & b \end{pmatrix}, \; \tilde{x}_k(i,j) = \begin{pmatrix} x_k(i,j) & & & 0 \\ & x_k(i,j) & & \\ & & \ddots & \\ 0 & & & x_k(i,j) \end{pmatrix}$$

in $M_m(A_0)$, and set

$$y_k = \sum_{i,j=1}^{n} \tilde{x}_k(i,j) e_{ij} \in M_m(A).$$

Then

$$y_k \, \tilde{b} \, y_k^* = \sum_{i,j,\alpha} \tilde{x}_k(i,\alpha) \, \tilde{b} \, \tilde{x}_k(j,\alpha)^* e_{ij} \to \tilde{a} \sum_{i=1}^{n} e_{ii}.$$

Since $\tilde{a}$ commutes with $M_m(A \cap A_0')$ and $\hat{e}$ is equivalent to e_{ii} in $M_m(A \cap A_0')$ we get

$$n\langle ae \rangle = n\langle \tilde{a}\hat{e} \rangle = \sum_{i=1}^{n} \langle \tilde{a} e_{ii} \rangle = \langle \tilde{a} \sum_{i=1}^{n} e_{ii} \rangle \leq \langle \tilde{b} \rangle = m\langle b \rangle.$$

□

Lemma 2.7. *Let A be a C^*–algebra, and let $(A_n)_1^\infty$ be an increasing sequence of C^*–subalgebras of A whose union is dense in A.*

(i) *For every non–zero positive element a in A there is an $n \in \mathbb{N}$ and a non–zero positive element b in A_n such that $\langle b \rangle \leq \langle a \rangle$ in $S(A)$.*

(ii) *If p is a projection in A_n and b is a positive element in A_n such that $\langle p \rangle \leq \langle b \rangle$ in $S(A)$, then $\langle p \rangle \leq \langle b \rangle$ in $S(A_m)$ for some $m \geq n$.*

Proof. (i) This follows easily from [**12**], 2.2 and 2.4.

(ii) If $\langle p \rangle \leq \langle b \rangle$ in $S(A)$, then $p = xbx^*$ for some x in A (by [**12**], 2.4). Find $m \geq n$ and $y \in A_m$ such that $\|yby^* - p\| < 1/2$. Then $p \precsim yby^*$ (relative to A_m) by [**12**], 2.2, whence $\langle p \rangle \leq \langle b \rangle$ in $S(A_m)$. □

Let in the following A and σ be as in Theorem 2.2. Set

$$A_n = \Big(\bigotimes_{j=1}^{n} B \Big) \otimes \mathbb{C}1 \otimes \mathbb{C}1 \otimes \cdots \quad \subseteq A.$$

Then $(A_n)_1^\infty$ is an increasing sequence of subalgebras of A whose union is dense in A.

Lemma 2.8. *There exists $m \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ for which $n\langle \sigma^k(1)\rangle \leq m\langle 1\rangle$ in $S(A)$.*

Proof. Observe first that $1-\sigma(1)$ $(= (1-p)\otimes 1 \otimes \cdots)$ is non–zero. Accordingly, $\langle 1-\sigma(1)\rangle$ is an order unit for $S(A)$, and so $\langle \sigma(1)\rangle \leq m\langle 1-\sigma(1)\rangle$ for some $m \in \mathbb{N}$. Hence $(m+1)\langle\sigma(1)\rangle \leq m\langle 1\rangle$, which again implies that $(m+1)\langle \sigma^k(1)\rangle \leq m\langle \sigma^{k-1}(1)\rangle$ for all $k \in \mathbb{N}$. We therefore have

$$j\langle \sigma^{k-1}(1)\rangle \geq (j+1)\langle \sigma^k(1)\rangle$$

for every $k \in \mathbb{N}$ and every $j \geq m$. Thus

$$m\langle 1\rangle \geq (m+1)\langle\sigma(1)\rangle \geq (m+2)\langle \sigma^2(1)\rangle \geq (m+3)\langle\sigma^3(1)\rangle \geq \cdots ,$$

from which the claim easily follows. □

Lemma 2.9. *For each non–zero $a \in A^+$ there are integers $k, n \geq 1$ such that $n\langle\sigma^k(1)\rangle \leq n\langle a\rangle$.*

Proof. By Lemma 2.4 it suffices to show that there exists an integer k such that $d(\langle \sigma^k(1)\rangle) < d(\langle a\rangle)$ for all $d \in DF(A)$. Put

$$c = \inf\{d(\langle a\rangle) \mid d \in DF(A)\} > 0,$$

let $m \in \mathbb{N}$ be as in Lemma 2.8 and find $n \in \mathbb{N}$ such that $m/n < c$. Then by Lemma 2.8 there is $k \in \mathbb{N}$ such that $n\langle \sigma^k(1)\rangle \leq m\langle 1\rangle$, and so $d(\langle\sigma^k(1)\rangle) < c \leq d(\langle a\rangle)$ for all $d \in DF(A)$. □

Proof of Theorem 2.2. By Theorem 2.1 (ii) and Lemma 2.3 it suffices to show that for each non–zero positive a in A there is a $k \in \mathbb{N}$ such that $\langle\sigma^k(1)\rangle \leq \langle a\rangle$.

Let $m \in \mathbb{N}$ be as in Lemma 2.8 and use Lemma 2.5 to find a non–zero positive b in A with $m\langle b\rangle \leq \langle a\rangle$. Use Lemma 2.7 (i) to find $l \in \mathbb{N}$ and a non–zero positive element b_1 in A_l with $\langle b_1\rangle \leq \langle b\rangle$. According to Lemma 2.9 and the choice of m there exist $k_1, k_2, n \in \mathbb{N}$ such that $n\langle\sigma^{k_1}(1)\rangle \leq n\langle b_1\rangle$ and $n\langle \sigma^{k_2}(1)\rangle \leq m\langle 1\rangle$. Use Lemma 2.7 (ii) to find $j \geq \max\{l, k_1\}$ such that the inequality $n\langle\sigma^{k_1}(1)\rangle \leq n\langle b_1\rangle$ holds in $S(A_j)$. Let λ be the j'th power of the one–sided Bernoulli–shift on A. Then λ is an endomorphism on A whose image is equal to $A \cap A_j'$. Observe that $n\langle\lambda(\sigma^{k_2}(1))\rangle \leq m\langle 1\rangle$ in $S(A\cap A_j')$. It follows that there exists a system of matrix units $(e_{ij})_{1\leq i,j\leq n}$ in $M_m(A\cap A_j')$ such that e_{11} is equivalent to

$$\hat{e} = \begin{pmatrix} \lambda(\sigma^{k_2}(1)) & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} \in M_m(A \cap A_j')$$

relative to $M_m(A \cap A'_j)$. Set $k = j + k_2$ ($\geq k_1 + k_2$). Then Lemma 2.6 yields

$$\langle \sigma^k(1) \rangle \leq \langle \lambda(\sigma^{k_2}(1))\sigma^{k_1}(1) \rangle \leq n \langle \lambda(\sigma^{k_2}(1))\sigma^{k_1}(1) \rangle \leq m \langle b_1 \rangle \leq \langle a \rangle$$

as desired. □

§3. Purely infinite simple free product C^*–algebras

In this section, we use Theorem 2.1 to show that certain C^*–algebras arising as reduced free products are purely infinite and simple.

For any C^*–algebra A with state φ, denote the defining mapping $A \to L^2(A, \varphi)$ by $a \mapsto \hat{a}$. For a Hilbert space, $\mathcal{H}$, we denote by $\mathcal{K}(\mathcal{H})$ the C^*–algebra of compact operators on $\mathcal{H}$.

Theorem 3.1. *Let $A \neq \mathbb{C}$ and B be C^*–algebras with states φ_A and φ_B, respectively, whose G.N.S. representations are faithful. Fix $N \in \{2, 3, 4, \dots\}$, let $(e_{ij})_{1 \leq i,j \leq N}$ be a system of matrix units for $M_N(\mathbb{C})$ and let φ_N denote the state on $M_N(\mathbb{C})$ such that $\varphi_N(e_{11}) = 1$. Consider the reduced free product C^*–algebra,*

$$(\mathfrak{A}, \varphi) = (A, \varphi_A) * (M_N(\mathbb{C}) \otimes B, \varphi_N \otimes \varphi_B).$$

If the pair $\big((A, \varphi_A), (B, \varphi_B)\big)$ has property Q, defined below, then $\mathfrak{A}$ is simple and purely infinite.

Several of the intermediate results in the proof of this theorem are valid also without assuming property Q; we will explicitly remark that we need property Q whenever this is the case. Let us now define property Q. Write

$$\mathcal{H}_A = L^2(A, \varphi_A), \quad \overset{o}{\mathcal{H}}_A = \mathcal{H}_A \ominus \mathbb{C}\hat{1},$$
$$\mathcal{H}_B = L^2(B, \varphi_B), \quad \overset{o}{\mathcal{H}}_B = \mathcal{H}_B \ominus \mathbb{C}\hat{1},$$

and let λ_A and λ_B denote the left actions of A on $\mathcal{H}_A$, respectively B on $\mathcal{H}_B$. Denote by $A *_r B$ the reduced C^*–algebra free product,

$$(A *_r B, \varphi_{A*B}) = (A, \varphi_A) * (B, \varphi_B).$$

Let $\mathcal{H}_{A*B} = L^2(A *_r B, \varphi_{A*B})$ and denote the usual left action (see [**14**, §1.5]) of $A *_r B$ on $\mathcal{H}_{A*B}$ by λ_{A*B}. We have

$$\mathcal{H}_{A*B} = \mathbb{C}\hat{1} \oplus \bigoplus_{\substack{n \geq 1 \\ X_j \in \{A,B\} \\ X_j \neq X_{j+1}}} \overset{o}{\mathcal{H}}_{X_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{X_n}.$$

Define the following subsets of $\mathcal{H}_{A*B}$:

$$\begin{aligned}
\mathcal{F} &= \overset{o}{\mathcal{H}}_A \oplus \bigoplus_{n\geq 1} \overset{o}{\mathcal{H}}_A \otimes (\overset{o}{\mathcal{H}}_B \otimes \overset{o}{\mathcal{H}}_A)^{\otimes n} \\
\mathcal{F}_l &= \mathcal{F} \oplus (\overset{o}{\mathcal{H}}_B \otimes \mathcal{F}) \\
\mathcal{F}_r &= \mathcal{F} \oplus (\mathcal{F} \otimes \overset{o}{\mathcal{H}}_B).
\end{aligned}$$

Identify $\mathcal{H}_B$ with the subspace, $\mathbb{C}\hat{1} \oplus \overset{o}{\mathcal{H}}_B \subseteq \mathcal{H}_{A*B}$. Let V be the isometry from $\mathcal{H}_{A*B} \ominus \mathcal{H}_B$ onto $\mathcal{F}_l \otimes \mathcal{H}_B$ that sends $\overset{o}{\mathcal{H}}_{X_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{X_n}$ to $(\overset{o}{\mathcal{H}}_{X_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{X_n}) \otimes \hat{1}$ if $X_n = A$ and to $(\overset{o}{\mathcal{H}}_{X_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{X_{n-1}}) \otimes \overset{o}{\mathcal{H}}_B$ if $X_n = B$.

Definition 3.2. Let p_B be the orthogonal projection from $\mathcal{H}_{A*B}$ onto $\mathcal{H}_B$. We say that the pair $\big((A,\varphi_A),(B,\varphi_B)\big)$ has *property Q* if

$$V(1-p_B)\lambda_{A*B}(A *_r B)(1-p_B)V^* \ \bigcap\ \big(\mathcal{K}(\mathcal{F}_l) \otimes B(\mathcal{H}_B)\big) = \{0\}.$$

Proposition 3.3. *$\big((A,\varphi_A),(B,\varphi_B)\big)$ has property Q if any of the following conditions are satisfied:*

(i) *$(A,\varphi_A) = (C_r^*(G_1),\tau_{G_1})$ and $(B,\varphi_B) = (C_r^*(G_2),\tau_{G_2})$ where G_1 and G_2 are nontrivial discrete groups and τ_{G_ι} is the canonical trace on $C_r^*(G_\iota)$;*

(ii) *there are unitaries $u_A \in B(\mathcal{H}_A) \cap \lambda_A(A)'$ and $u_B \in B(\mathcal{H}_B) \cap \lambda_B(B)'$ such that $u_A\hat{1}_A \perp \hat{1}_A$ and $u_B\hat{1}_B \perp \hat{1}_B$;*

(iii) *$B = \mathbb{C}$ and $\lambda_A(A) \cap \mathcal{K}(\mathcal{H}_A) = \{0\}$;*

(iv) *B is finite dimensional and $\lambda_{A*B}(A *_r B) \cap \mathcal{K}(\mathcal{H}_{A*B}) = \{0\}$.*

Proof. It is clear that (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ property Q. Since the right translation operators on $l^2(G_\iota)$ are unitaries commuting with the left translation operators, if (A,φ_A) and (B,φ_B) are as in condition (i) then condition (ii) is satisfied. Hence we must only show that condition (ii) implies property Q. Assume (ii) is satisfied and let $0 \neq x \in A *_r B$, $x \geq 0$. We will show that $V(1-p_B)\lambda_{A*B}(x)(1-p_B)V^* \notin \mathcal{K}(\mathcal{F}_l)\otimes B(\mathcal{H}_B)$. There is $\zeta \in \mathcal{H}_{A*B}$ such that $\langle \lambda_{A*B}(x)\zeta,\zeta\rangle \neq 0$ and we may assume without loss of generality that either

$$\left.\begin{aligned}
&\zeta = \hat{1}, \\
\text{or } &\zeta \in \overset{o}{\mathcal{H}}_B, \\
\text{or } &\zeta \in (\overset{o}{\mathcal{H}}_A \otimes \overset{o}{\mathcal{H}}_B)^{\otimes n},\ \text{some } n \geq 1, \\
\text{or } &\zeta \in \overset{o}{\mathcal{H}}_B \otimes (\overset{o}{\mathcal{H}}_A \otimes \overset{o}{\mathcal{H}}_B)^{\otimes n},\ \text{some } n \geq 1,
\end{aligned}\right\} \tag{2}$$

or

$$\left.\begin{aligned} &\zeta \in \overset{o}{\mathcal{H}}_A, \\ \text{or } &\zeta \in (\overset{o}{\mathcal{H}}_B \otimes \overset{o}{\mathcal{H}}_A)^{\otimes n},\ \text{some } n \geq 1, \\ \text{or } &\zeta \in \overset{o}{\mathcal{H}}_A \otimes (\overset{o}{\mathcal{H}}_B \otimes \overset{o}{\mathcal{H}}_A)^{\otimes n},\ \text{some } n \geq 1. \end{aligned}\right\} \tag{3}$$

Let

$$\sigma_A : B(\mathcal{H}_A) \to B(\mathcal{H}_{A*B}),$$
$$\sigma_B : B(\mathcal{H}_B) \to B(\mathcal{H}_{A*B})$$

be the "right actions" as in [**14**, §1.6], so that, by Voiculescu's characterization of the commutant, $\sigma_A(u_A)$, $\sigma_B(u_B) \in \lambda_{A*B}(A *_r B)' \cap B(\mathcal{H}_{A*B})$. If one of the four cases in (2) holds then let $w_m = (\sigma_B(u_B)\sigma_A(u_A))^m$ so that, *e.g.*,

$$w_m(\overset{o}{\mathcal{H}}_A \otimes \overset{o}{\mathcal{H}}_B)^{\otimes n} \subseteq (\overset{o}{\mathcal{H}}_A \otimes \overset{o}{\mathcal{H}}_B)^{\otimes n+m}.$$

On the other hand, if one of the three cases in (3) holds then let $w_m = (\sigma_A(u_A)\sigma_B(u_B))^m$. Then for every $m \geq 1$, $V w_m \zeta \in (P_{k_m}\mathcal{F}_l) \otimes \mathcal{H}_B$, where P_k is the projection from $\mathcal{F}_l$ onto $(\overset{o}{\mathcal{H}}_B \otimes \overset{o}{\mathcal{H}}_A)^{\otimes k/2}$ if k is even and onto $\overset{o}{\mathcal{H}}_A \otimes (\overset{o}{\mathcal{H}}_B \otimes \overset{o}{\mathcal{H}}_A)^{\otimes (k-1)/2}$ if k is odd, and where $k_{j+1} = k_j + 2$ for all $j \geq 1$. In particular, $w_1\zeta$, $w_2\zeta$, $w_3\zeta, \ldots$ is a sequence of mutually orthogonal vectors, all having the same norm. But because w_m is a unitary that commutes with $\lambda_{A*B}(x)$, $\langle \lambda_{A*B}(x) w_m\zeta, w_m\zeta\rangle = \langle \lambda_{A*B}(x)\zeta, \zeta\rangle \neq 0$ for all m, hence $V(1-p_B)\lambda_{A*B}(x)(1-p_B)V^* \notin \mathcal{K}(\mathcal{F}_l) \otimes B(\mathcal{H}_B)$. □

Now we prove Theorem 3.1 and we begin by examining $L^2(\mathfrak{A}, \varphi)$. Note that $(\hat{e}_{n1})_{1\leq n\leq N}$ is an orthonormal basis for $L^2(M_N(\mathbb{C}), \varphi_N)$ and that $\hat{e}_{11} = \hat{1}$. Thus

$$\mathcal{H}_{M_N(\mathbb{C})\otimes B} \overset{\text{def}}{=} L^2(M_N(\mathbb{C}) \otimes B, \varphi_N \otimes \varphi_B) = \bigoplus_{n=1}^{N} \hat{e}_{n1} \otimes \mathcal{H}_B.$$

Let $\overset{o}{\mathcal{H}}_{M_N(\mathbb{C})\otimes B} = \mathcal{H}_{M_N(\mathbb{C})\otimes B} \ominus \mathbb{C}(\hat{e}_{11} \otimes \hat{1}_B)$. By Voiculescu's construction,

$$\mathcal{H} \overset{\text{def}}{=} L^2(\mathfrak{A}, \varphi) = \mathbb{C}\hat{1} \oplus \bigoplus_{\substack{n\geq 1 \\ X_j \in \{A, M_N(\mathbb{C})\otimes B\} \\ X_j \neq X_{j+1}}} \overset{o}{\mathcal{H}}_{X_1} \otimes \cdots \otimes \overset{o}{\mathcal{H}}_{X_n},$$

with $\mathfrak{A}$ acting on $\mathcal{H}$ on the left in the usual way. We will examine how $e_{11}\mathfrak{A}e_{11}$ acts on $e_{11}\mathcal{H}$. Identify $\mathcal{H}_B$ with $\hat{e}_{11} \otimes \mathcal{H}_B \subseteq \mathcal{H}_{M_n(\mathbb{C})\otimes B}$ and $\overset{o}{\mathcal{H}}_B$ with $\hat{e}_{11} \otimes \overset{o}{\mathcal{H}}_B$ and thus identify $\mathcal{H}_{A*B}$ with

the subspace of $e_{11}\mathcal{H} \subseteq \mathcal{H}$ spanned by all the tensors in $\overset{o}{\mathcal{H}}_A$ and $\hat{e}_{11} \otimes \overset{o}{\mathcal{H}}_B$. Consider the subspaces of $e_{11}\mathcal{H}$ defined by

$$\begin{aligned}
\mathcal{V}_0 &= \mathcal{H}_{A*B} \subseteq \mathcal{H} \\
\mathcal{V}_{[n]} &= \bigoplus_{2\le k_1,\dots,k_n\le N} \mathcal{F}_l \otimes (\mathcal{H}_B \otimes \hat{e}_{k_1 1}) \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_2 1}) \otimes \cdots \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_n 1}) \\
\mathcal{V}_{(n)} &= \bigoplus_{2\le k_1,\dots,k_n\le N} \mathcal{F}_l \otimes (\mathcal{H}_B \otimes \hat{e}_{k_1 1}) \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_2 1}) \otimes \cdots \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_n 1}) \otimes \mathcal{F}_r.
\end{aligned}$$

Then

$$e_{11}\mathcal{H} = \mathcal{V}_0 \bigoplus_{n\ge 1} (\mathcal{V}_{[n]} \oplus \mathcal{V}_{(n)}).$$

Let $\mathcal{W}_n = \bigoplus_{k\ge n}(\mathcal{V}_{(k)} \oplus \mathcal{V}_{[k]})$ so that $\bigcap_{n\ge 1} \mathcal{W}_n = \{0\}$.

We also regard $A *_r B$ as a subalgebra of $\mathfrak{A}$ in the canonical way. Let $A_0 \subseteq \ker\varphi_A$ be such that spanA_0 is norm dense in $\ker\varphi_A$ and $\{\hat{x} \mid x \in A_0\}$ is an orthonormal basis for $\overset{o}{\mathcal{H}}_A$ and let $B_0 \subseteq \ker\varphi_B$ be such that spanB_0 is norm dense in $\ker\varphi_B$ and $\{\hat{x} \mid x \in B_0\}$ is an orthonormal basis for $\overset{o}{\mathcal{H}}_B$. Let

$$\begin{aligned}
F &= \bigcup_{n\ge 1} A_0 \underbrace{B_0 A_0 \cdots B_0 A_0}_{n-1 \text{ times } B_0 A_0} \\
F_l &= F \cup B_0 F,
\end{aligned}$$

so that $\{\hat{x} \mid x \in F\}$ is an orthonormal basis for $\mathcal{F}$, $\{\hat{x} \mid x \in F_l\}$ is an orthonormal basis for $\mathcal{F}_l$ and span$(\{1\} \cup B_0 \cup F_l \cup F B_0)$ is dense in $A *_r B$. For $x \in F_l$ and $2 \le n \le N$ let

$$T(x, n) = e_{11} x e_{n1} \in \mathfrak{A}.$$

Then $T(x, n)$ is maps $e_{11}\mathcal{H}$ onto

$$\begin{aligned}
&\Big(\hat{x}\otimes(\mathcal{H}_B \otimes \hat{e}_{n1})\Big) \quad\oplus\quad \Big(\hat{x} \otimes (\mathcal{H}_B \otimes \hat{e}_{n1}) \otimes \mathcal{F}_r\Big) \quad\oplus \\
&\quad\oplus \bigoplus_{\substack{m\ge 1 \\ 2\le k_1,\dots,k_m\le N}} \hat{x} \otimes (\mathcal{H}_B \otimes \hat{e}_{n1}) \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_2 1}) \otimes \cdots \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_m 1}) \\
&\quad\oplus \bigoplus_{\substack{m\ge 1 \\ 2\le k_1,\dots,k_m\le N}} \hat{x} \otimes (\mathcal{H}_B \otimes \hat{e}_{n1}) \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_2 1}) \otimes \cdots \otimes \mathcal{F} \otimes (\mathcal{H}_B \otimes \hat{e}_{k_m 1}) \otimes \mathcal{F}_r.
\end{aligned} \tag{4}$$

More specifically, taking an orthonormal basis for $e_{11}\mathcal{H}$ consisting of tensors, each of the form $\hat{y}$ or $\hat{y} \otimes \cdots$ for some $y \in F_l$, we see that $T(x, n)$ maps each such element to $\hat{x} \otimes \hat{e}_{1n} \otimes$

$\hat{y}(\otimes\cdots)$. Thus $T(x,n)$ is a one–to–one mapping from an orthonormal basis for $e_{11}\mathcal{H}$ onto and orthonormal set spanning the space given in (4), thus is an isometry from $e_{11}\mathcal{H}$ onto this space. Hence $(T(x,n))_{x\in F_l,\,2\le n\le N}$ is a family of isometries having orthogonal ranges. Moreover, the strong–operator limit

$$P_{\mathcal{W}_n} = \sum_{\substack{x_1,\dots,x_n\in F_l\\ 2\le k_1,\dots,k_n\le N}} T(x_1,k_1)\cdots T(x_n,k_n)T(x_n,k_n)^*\cdots T(x_1,k_1)^* \tag{5}$$

is the projection from $e_{11}\mathcal{H}$ onto $\mathcal{W}_n$.

Lemma 3.4. *For every $x\in F_l$ and every $2\le n,m\le N$,*

$$e_{nn}xe_{mm}=0.$$

Proof. We have

$$e_{mm}\mathcal{H} = (\mathcal{H}_B\otimes\hat{e}_{m1}) \quad\oplus\quad (\mathcal{H}_B\otimes\hat{e}_{m1})\otimes\mathcal{F}_r \quad\oplus\quad \cdots,$$

where this formula continues as in (4), but without the $\hat{x}$, so $xe_{mm}\mathcal{H}=T(x,m)\mathcal{H}\subseteq e_{11}\mathcal{H}$. □

Lemma 3.5. *If $z_1,z_2\in A*_r B$ then*

$$e_{11}z_1e_{11}z_2e_{11}\in e_{11}z_1z_2e_{11}+\overline{\mathrm{span}}\bigcup_{\substack{2\le n\le N\\ x,y\in F_l}} T(x,n)BT(y,n)^*. \tag{6}$$

Proof. It will be enough to show (6) for z_1 and z_2 in a set that densely spans $A*_r B$, hence we assume without loss of generality that $z_j\in B\cup F_lB$. If either $z_1\in B$ or $z_2\in B$ then $e_{11}z_1e_{11}z_2e_{11}=e_{11}z_1z_2e_{11}$. If $z_j=f_jb_j$ for $f_j\in F_l$ and $b_j\in B$, $(j=1,2)$, then

$$\begin{aligned}
e_{11}z_1e_{11}z_2^*e_{11} &= e_{11}z_1z_2^*e_{11}-\sum_{n=2}^N e_{11}z_1e_{n1}e_{1n}z_2^*e_{11}\\
&= e_{11}z_1z_2^*e_{11}-\sum_{n=2}^N e_{11}f_1e_{n1}b_1b_2^*e_{1n}f_2^*e_{11}\\
&= e_{11}z_1z_2^*e_{11}-\sum_{n=2}^N T(f_1,n)b_1b_2^*T(f_2,n)^*.
\end{aligned}$$

□

Lemma 3.6. *For every $x_0 \in F_l$, and $2 \le n \le N$,*

$$e_{11}(A *_r B)e_{11}T(x_0, n) \subseteq \overline{\operatorname{span}} \bigcup_{x \in F_l} T(x, n)B.$$

Proof. Let $z \in A *_r B$. Then

$$e_{11}ze_{11}T(x_0, n) = e_{11}ze_{11}x_0e_{n1} = e_{11}zx_0e_{n1} - \sum_{k=2}^{N} e_{11}ze_{kk}x_0e_{n1} = e_{11}zx_0e_{n1},$$

where the last equality follows from Lemma 3.4. Thus

$$e_{11}ze_{11}T(x_0, n) \in e_{11}(A *_r B)e_{n1} \subseteq \overline{\operatorname{span}} \bigcup_{x \in F_l} T(x, n)B.$$

□

Lemma 3.7.

$$\begin{aligned} e_{11}\mathfrak{A}e_{11} &= C^*(e_{11}Ae_{11} \cup e_{11}B \cup \bigcup_{2 \le n \le N} e_{11} \ker \varphi_A e_{n1}) \\ &= C^*(e_{11}(A *_r B)e_{11} \cup \bigcup_{2 \le n \le N} e_{11}F_le_{n1}) \\ &= \overline{\operatorname{span}}\Bigg(\bigcup_{\substack{p,q \ge 0 \\ x_j, y_j \in F_l \\ 2 \le n_j, m_j \le N}} T(x_1, n_1) \cdots T(x_p, n_p)e_{11}(A *_r B)e_{11}T(y_q, m_q)^* \cdots T(y_1, m_1)^* \Bigg). \end{aligned} \tag{7}$$

Proof. Because $\mathfrak{A}$ is generated by A and B together with the system of matrix units $(e_{ij})_{1 \le i,j \le N}$, we have that

$$e_{11}\mathfrak{A}e_{11} = C^*(\bigcup_{1 \le i,j \le N} e_{1i}Ae_{j1} \cup \bigcup_{1 \le i,j \le N} e_{1i}Be_{j1}).$$

Using Lemma 3.4 and that B commutes with e_{ij}, the first equality of (7) clearly holds and then the second equality is also clear. Now the third equality follows from Lemmas 3.5 and 3.6 and the fact that, for $x, y \in F_l$ and $2 \le n, m \le N$,

$$T(y, m)^*T(x, n) = \begin{cases} e_{11} & \text{if } x = y \text{ and } n = m \\ 0 & \text{otherwise.} \end{cases}$$

□

For $p \ge 1$ let

$$\begin{aligned} \mathfrak{B}_p = \overline{\operatorname{span}}\Bigg(& \bigcup_{\substack{0 \le k \le p-1 \\ x_j, y_j \in F_l \\ 2 \le n_j, m_j \le N}} T(x_1, n_1) \cdots T(x_k, n_k)e_{11}(A *_r B)e_{11}T(y_k, m_k)^* \cdots T(y_1, m_1)^* \\ & \cup \bigcup_{\substack{x_j, y_j \in F_l \\ 2 \le n_j, m_j \le N}} T(x_1, n_1) \cdots T(x_p, n_p)BT(y_p, m_p)^* \cdots T(y_1, m_1)^* \Bigg) \end{aligned}$$

and let $\mathfrak{B} = \overline{\bigcup_{p \ge 1} \mathfrak{B}_p}$. Hence $\mathcal{W}_n$ is an invariant subspace of $\mathfrak{B}$, for all $n \ge 1$. Based on Lemmas 3.5 and 3.6, we have

Observation 3.8. Each $\mathfrak{B}_p$ and also $\mathfrak{B}$ is a C^*–subalgebra of $e_{11}\mathfrak{A}e_{11}$. For any fixed $x_0 \in F_l$, $e_{11}\mathfrak{A}e_{11} = C^*(\mathfrak{B} \cup \{T(x_0, 2)\})$ and $z \mapsto T(x_0, 2)zT(x_0, 2)^*$ is an endomorphism, call it σ, of $\mathfrak{B}$. Hence $e_{11}\mathfrak{A}e_{11}$ is a quotient of the universal crossed product, $\mathfrak{B} \rtimes_\sigma \mathbb{N}$, of $\mathfrak{B}$ by the endomorphism σ. Therefore, once we have proved the following two propositions, Theorem 2.1 will imply that this universal crossed product is simple and purely infinite and Theorem 3.1 will be proved.

Proposition 3.9. *If $\big((A, \varphi_A), (B, \varphi_B)\big)$ has property Q then for every $z \in \mathfrak{B}$ such that $z \geq 0$ and $z \neq 0$, there are $n \in \mathbb{N}$ and $x \in \mathfrak{B}$ such that $xzx^* = \sigma^n(e_{11})$, i.e. $\sigma^n(e_{11}) \lesssim z$.*

Proposition 3.10. *Let $(\overline{\mathfrak{B}}, \alpha)$ be the C^*–dynamical system associated to $(\mathfrak{B}, \sigma)$ as described at the beginning of §2. For no $m \geq 1$ is the automorphism α^m of $\overline{\mathfrak{B}}$ inner.*

In order to prove these propositions, let us define, for $n \geq 1$,

$$\mathcal{I}_n = \overline{\operatorname{span}}\Bigg(\bigcup_{\substack{k \geq n \\ x_j, y_j \in F_l \\ 2 \leq n_j, m_j \leq N}} T(x_1, n_1) \cdots T(x_k, n_k) e_{11} (A *_r B) e_{11} T(y_k, m_k)^* \cdots T(y_1, m_1)^* \Bigg).$$

Clearly, $\mathcal{I}_n$ is an ideal of $\mathfrak{B}$, $\mathcal{I}_n \supseteq \mathcal{I}_{n+1}$, $\bigcap_{n \geq 1} \mathcal{I}_n = \{0\}$ and $\mathcal{I}_n$ vanishes on $\mathcal{H} \ominus \mathcal{W}_n$. The following lemma may be of interest, although it is not used in the sequel.

Lemma 3.11. *$\mathfrak{B}/\mathcal{I}_1 \cong A *_r B$ and*

$$\mathcal{I}_n/\mathcal{I}_{n+1} \cong (A *_r B) \otimes \mathcal{K} \quad \textit{for all } n \geq 1, \tag{8}$$

where $\mathcal{K}$ appearing in (8) is the algebra of compact operators on the infinite dimensional Hilbert space $((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes n})$.

Proof. Let us first consider the case of $\mathfrak{B}/\mathcal{I}_1$. Let $\iota : A *_r B \to \mathfrak{A}$ be the canonical (functorial) inclusion. Consider the mapping $\pi : A *_r B \to \mathfrak{B}/\mathcal{I}_1$ given by $\pi(z) = [e_{11}\iota(z)e_{11}] \in \mathfrak{B}/\mathcal{I}_1$. By Lemma 3.5, π is a $*$–homomorphism. But since $\mathcal{I}_1$ vanishes on $\mathcal{V}_0 = \mathcal{H}_{A*B}$, the map from $\mathfrak{B}/\mathcal{I}_1$ to $A *_r B$ given by $[y] \mapsto y|_{\mathcal{V}_0}$ is well–defined and is the inverse of π. Hence π is an isomorphism.

For $n \geq 1$,

$$\{T(x_1, k_1) \cdots T(x_n, k_n) T(y_n, l_n)^* \cdots T(y_1, l_1)^* \mid x_j, y_j \in F_l,\ 2 \leq k_j, l_j \leq N\}$$

is a system of matrix units whose closed linear span is a copy of $\mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes n})$ and there is an isomorphism $\mathcal{I}_n \to \mathfrak{B} \otimes \mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes n})$ given by, for $z \in A *_r B$ and $p \geq n$,

$$\begin{aligned} &T(x_1,k_1)\cdots T(x_p,k_p)zT(y_p,l_p)^*\cdots T(y_1,l_1)^* \mapsto \\ &\quad\mapsto \big(T(x_{n+1},k_{n+1})\cdots T(x_p,k_p)e_{11}ze_{11}T(y_p,l_p)^*\cdots T(y_{n+1},l_{n+1})^*\otimes \\ &\qquad\qquad\qquad \otimes T(x_1,k_1)\cdots T(x_n,k_n)T(y_n,l_n)^*\cdots T(y_1,l_1)^*\big). \end{aligned}$$

This isomorphism sends $\mathcal{I}_{n+1}$ onto $\mathcal{I}_1 \otimes \mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes n})$, so

$$\mathcal{I}_n/\mathcal{I}_{n+1} \cong (\mathfrak{B}/\mathcal{I}_1) \otimes \mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes n})$$

. □

Lemma 3.12. *If $\big((A,\varphi_A),(B,\varphi_B)\big)$ has property Q, $z \in \mathfrak{B}$ and z vanishes on $\mathcal{W}_n$ for some n, then $z = 0$. Consequently $||P_{\mathcal{W}_n} z P_{\mathcal{W}_n}|| = ||z||$ for all $z \in \mathfrak{B}$ and for all $n \geq 1$.*

Proof. First consider the case $n = 1$. We have $e_{11}\mathcal{H} = \mathcal{V}_0 \oplus \mathcal{W}_1$ and there is a unitary

$$U_1 : \mathcal{W}_1 \to \mathcal{F}_l \otimes \mathbb{C}^{N-1} \otimes \mathcal{H}_B \otimes \mathcal{Y}$$

for an infinite dimensional Hilbert space, $\mathcal{Y}$, such that for $x, y \in F_l$ and $2 \leq n, m \leq N$,

$$T(x,n)T(y,m)^* = 0_{\mathcal{V}_0} \oplus U_1^*(f_{x,y} \otimes e_{n,m} \otimes 1_{\mathcal{H}_B} \otimes 1_{\mathcal{Y}})U_1$$

where $f_{x,y}$ is the rank–one operator on $\mathcal{F}_l$ sending $\hat{y}$ to $\hat{x}$ and where $e_{n,m}$ is the rank–one operator on $\mathbb{C}^{N-1}$ sending the $(m-1)$st standard basis vector to the $(n-1)$st. Hence we have that

$$\mathcal{I}_1 \subseteq 0_{\mathcal{V}_0} \oplus U_1^*\big(\mathcal{K}(\mathcal{F}_l) \otimes B(\mathbb{C}^{N-1}) \otimes B(\mathcal{H}_B) \otimes B(\mathcal{Y})\big)U_1. \tag{9}$$

For $d \in A *_r B$ we have $e_{11}de_{11}|_{\mathcal{V}_0} = \lambda_{A*B}(d)$ under the identification of $\mathcal{V}_0$ with $\mathcal{H}_{A*B}$. Recall (from just before Definition 3.2) the unitary

$$V : \mathcal{H}_{A*B} \ominus \mathcal{H}_B \to \mathcal{F}_l \otimes \mathcal{H}_B.$$

Let

$$\widetilde{V} : (\mathcal{H}_{A*B} \ominus \mathcal{H}_B) \otimes \mathbb{C}^{N-1} \to \mathcal{F}_l \otimes \mathbb{C}^{N-1} \otimes \mathcal{H}_B$$

be such that $\widetilde{V}^*(\xi_1 \otimes \xi_2 \otimes \xi_3) = V^*(\xi_1 \otimes \xi_3) \otimes \xi_2$, *i.e.* $\widetilde{V}$ just pushes $\mathbb{C}^{N-1}$ through and acts like V on the rest. Then it is easily seen for $d \in A *_r B$ that

$$U_1\big(e_{11}de_{11}|_{\mathcal{W}_1}\big)U_1^* = \widetilde{V}\big((1-p_B)\lambda_{A*B}(d)(1-p_B) \otimes 1_{\mathbb{C}^{N-1}}\big)\widetilde{V}^* \otimes 1_{\mathcal{Y}}. \tag{10}$$

From the proof of Lemma 3.11, there is a $*$–homomorphism $\psi : \mathfrak{B} \to A *_r B$, whose kernel is $\mathcal{I}_1$, given by $\psi(z) = z|_{\mathcal{V}_0}$ and the mapping $A *_r B \ni d \mapsto e_{11} d e_{11}$ is a right inverse for ψ. Hence $z = e_{11}\psi(z)e_{11} + z_1$ where $z_1 \in \mathcal{I}_1$ and so, using (9) and (10),

$$U_1(z|_{\mathcal{W}_1})U_1^* \in \Big(\widetilde{V}\big((1-p_B)\lambda_{A*B}(\psi(z))(1-p_B)\otimes 1_{\mathbb{C}^{N-1}}\big)\widetilde{V}^* + \mathcal{K}(\mathcal{F}_l)\otimes B(\mathbb{C}^{N-1})\otimes B(\mathcal{H}_B)\Big)\otimes B(\mathcal{Y}). \tag{11}$$

But assuming that property Q holds, using (11) the supposition that $z|_{\mathcal{W}_1} = 0$ implies that $\psi(z) = 0$. But then $z \in \mathcal{I}_1$ and $\mathcal{I}_1$ has support equal to $\mathcal{W}_1$, so $z = 0$. Hence the first part of the lemma is proved in the case $n = 1$.

Now we will show, for $n \geq 1$, and $z \in \mathfrak{B}$ that $z|_{\mathcal{W}_{n+1}} = 0$ implies $z|_{\mathcal{W}_n} = 0$, which when combined with the case proved above will show that $z = 0$. For every $x_1, \dots, x_n, y_1, \dots, y_n \in F_l$ and $2 \leq k_1, \dots, k_n, l_1, \dots, l_n \leq N$,

$$T(x_n, k_n)^* \cdots T(k_1, k_1)^* z T(y_1, l_1) \cdots T(y_n, l_n) \in \mathfrak{B}$$

vanishes on $\mathcal{W}_1$, hence is equal to zero. Therefore by (5) $P_{\mathcal{W}_n} z P_{\mathcal{W}_n} = 0$ and, since $\mathcal{W}_n$ is invariant under $\mathfrak{B}$, $zP_{\mathcal{W}_n} = 0$, as required.

Now since $P_{\mathcal{W}_n}$ commutes with $\mathfrak{B}$, the mapping $z \mapsto P_{\mathcal{W}_n} z P_{\mathcal{W}_n}$ is a $*$–homomorphism with zero kernel, proving that $||P_{\mathcal{W}_n} z P_{\mathcal{W}_n}|| = ||z||$. □

Proof of Proposition 3.9. First suppose $0 \neq z \geq 0$, $z \in \mathfrak{B}_p$, some $p \geq 1$. By Lemma 3.12, $z|_{\mathcal{W}_p} \neq 0$. Looking at the definition of $\mathcal{W}_p$, we see that there is a canonical unitary

$$U_p : \mathcal{W}_p \to (\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes p} \otimes \mathcal{H}_B \otimes \mathcal{Y}_p$$

for $\mathcal{Y}_p$ the infinite dimensional Hilbert space, $\mathbb{C}\xi \oplus \mathcal{F}_r \oplus \cdots$, such that

$$U_p \mathfrak{B}_p U_p^* = \mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes p}) \otimes \lambda_B(B) \otimes 1_{\mathcal{Y}_p}$$

with

$$(U_p T(x_1, k_1) \cdots T(x_n, k_n) T(y_n, l_n)^* \cdots T(y_1, l_1)^* U_p^*)_{x_j, y_j \in F_l,\, 2 \leq k_j, l_j \leq N}$$

being a system of matrix units for $\mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes p}) \otimes 1_{\mathcal{H}_B} \otimes 1_{\mathcal{Y}_p}$. Since $P_{\mathcal{W}_p}$ commutes with z we have $z \geq z|_{\mathcal{W}_p} = P_{\mathcal{W}_p} z P_{\mathcal{W}_p} \geq 0$ and there is a vector, $\zeta \in (\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes p} \otimes \mathcal{H}_B$ such that if q is the projection onto $\zeta \otimes \mathcal{Y}_p$ then $z \geq c U_p^* q U_p$ for some $c > 0$. Let $\zeta' \in \mathcal{F} \otimes \mathbb{C}^{N-1}$ be nonzero, so that $\zeta \otimes \zeta' \in (\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes p+1}$ and let q' be the projection onto $\zeta \otimes \zeta' \otimes \mathcal{H}_B \otimes \mathcal{Y}_{p+1}$. Then $z \geq c U_p^* q U_p^* \geq c U_{p+1}^* q' U_{p+1}$. Since both q' and $U_{p+1} T(x_0, 2)^{p+1} (T(x_0, 2)^*)^{p+1} U_{p+1}^*$ are

minimal projections in $\mathcal{K}((\mathcal{F}_l \otimes \mathbb{C}^{N-1})^{\otimes p}) \otimes 1_{\mathcal{H}_B} \otimes 1_{\mathcal{Y}_p} \subseteq U_{p+1}\mathfrak{B}_{p+1}U_{p+1}^*$, it is clear that $U_{p+1}^* q' U_{p+1}$ is equivalent in $\mathfrak{B}_{p+1}$ to the projection $T(x_0,2)^{p+1}(T(x_0,2)^*)^{p+1} = \sigma^{p+1}(e_{11})$. Hence $z \gtrsim \sigma^{p+1}(e_{11})$, as required, and there is $y \in \mathfrak{B}$ such that $\sigma^{p+1}(e_{11}) = yzy^*$. Since we may assume that $U_p^* \zeta U_p$ is in the range of the spectral projection for z corresponding to the interval $\left[\frac{1}{2}||z||, ||z||\right]$, we may assume $||y|| \le \sqrt{2}||z||^{-1/2}$.

For general $z \in \mathfrak{B}$, $0 \neq z \geq 0$, there is $p \geq 1$ and $\tilde{z} \in \mathfrak{B}_p$, $0 \neq \tilde{z} \geq 0$ such that $||z - \tilde{z}|| < ||z||/3$. Let y be such that $||y|| \le \sqrt{2}||\tilde{z}||^{-1/2}$ and $\sigma^{p+1}(e_{11}) = y\tilde{z}y^*$ holds. Then

$$||yzy^* - y\tilde{z}y^*|| \le \frac{2||z-\tilde{z}||}{||\tilde{z}||} \le \frac{||z||}{3||\tilde{z}||} < 1,$$

and by the continuous function calculus and the theory of projections there is $y' \in \mathfrak{B}$ such that $\sigma^{p+1}(e_{11}) = y'z(y')^*$, namely $\sigma^{p+1}(e_{11}) \lesssim z$. □

Proof of Proposition 3.10. Recall that $\overline{\mathfrak{B}}$ is the inductive limit of

$$\mathfrak{B} \xrightarrow{\sigma} \mathfrak{B} \xrightarrow{\sigma} \cdots$$

with corresponding embeddings $\mu_n : \mathfrak{B} \to \overline{\mathfrak{B}}$ $(n \geq 1)$ such that $\mu_n(z) = \mu_{n+1}(\sigma(z))$ and the automorphism α is defined by $\alpha(\mu_n(z)) = \mu_n(\sigma(z))$ for all $n \geq 1$ and $z \in \mathfrak{B}$. For $k \in \mathbb{Z}$ consider

$$\overline{\mathcal{I}}_k = \overline{\bigcup_{n \geq \max(1,-k)} \mu_n(\mathcal{I}_{k+n})} \subseteq \overline{\mathfrak{B}}.$$

Then $\overline{\mathcal{I}}_k$ is an ideal of $\overline{\mathfrak{B}}$ and

$$\overline{\mathcal{I}}_k \supseteq \overline{\mathcal{I}}_{k+1}, \quad \bigcup_k \overline{\mathcal{I}}_k = \overline{\mathfrak{B}}, \quad \bigcap_k \overline{\mathcal{I}}_k = \{0\} \quad \text{and} \quad \alpha(\overline{\mathcal{I}}_k) = \overline{\mathcal{I}}_{k+1}.$$

Let $\pi_k : \overline{\mathfrak{B}} \to \overline{\mathfrak{B}}/\overline{\mathcal{I}}_k$ be the quotient map $z \mapsto z + \overline{\mathcal{I}}_k$. Since the union of the ideals is $\overline{\mathfrak{B}}$ and their intersection is $\{0\}$, we have for every $z \in \overline{\mathfrak{B}}$ that

$$\lim_{k \to -\infty} ||\pi_k(z)|| = 0 \quad \text{and} \quad \lim_{k \to +\infty} ||\pi_k(z)|| = ||z||.$$

Moreover,

$$||\pi_k(z)|| = ||z + \mathcal{I}_k|| = ||\alpha(z + \mathcal{I}_k)|| = ||\alpha(z) + \overline{\mathcal{I}}_{k+1}|| = ||\pi_{k+1}(\alpha(z))||.$$

Supposing to obtain a contradiction that α^m is inner for some $m \geq 1$, there is $0 \neq u \in \overline{\mathfrak{B}}$ such that $\alpha^m(u) = u$. There are $\epsilon > 0$ and $K_1, K_2 \in \mathbb{Z}$ such that $||\pi_k(u)|| < \epsilon$ for all $k \leq K_1$ and $||\pi_k(u)|| > 2\epsilon$ for all $k \geq K_2$. But $||\pi_k(u)|| = ||\pi_{k+m}(\alpha^m(u))|| = ||\pi_{k+m}(u)||$, which implies $\epsilon > 2\epsilon$. □

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